PROPAGATION OF CRACKS IN COMPRESSED BODIES

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The propagation of brittle cracks in compressed bodies is considered. In section 1, basic features of the strength theory of brittle bodies are considered in the idealized case of crack with free edges. In section 2 an effective closed solution of the plane elastic problem is obtained for 'closed' cracks distributed along a straight line. A 'closed' cracks is considered as a mathematical cut along which jumps in normal displacement, normal stress and shear stress (the latter in particular becoming equal to zero), are given. The interaction of forces between opposite edges of the cut may be completely arbitrary and nonlinear (in a linear case, dry Coulomb friction with coupling). The solution obtained is used in section 3 for a more exact consideration on the problem of strength of brittle bodies under compression. It is shown that the strength of compressed brittle bodies is fully determined by the presence of shear cracks and by some material constants which characterize the shear strength (basically by a so-called [1] shear modulus of cohesion). In section 4 an independence law is stated which implies that the growth direction of an arbitrary 'closed' crack and the nature of fracturing is fully defined by material properties at the crack tip. At the same time, the angle between the original direction of the 'closed' crack and the deviating crack assumes a 'quantised' value: 0° or 75°. An application is shown in section 5; a theoretical picture of a rock burst is considered and some conclusions are derived regarding the safest shape of excavations. The present state of the rock burst problem and the degree of qualitative understanding of this effect is discussed in the monograph of Avershin [2] and partly in the book of Khodot [3]. Basic ideas and hypotheses of the crack theory, as developed by Barenblatt (see a review [4]), have been employed in the present work. It should be noted that literature dealing with cracks in compressed bodies is practically nonexistent; the only recent paper [5] has a number of shortcomings.

1. An improved theory of strength of compressed brittle bodies. In accordance with Griffith's ideas, a real brittle body has many defects or microcracks, where the surface energy is accumulated. Then, fracture of the body in tension is explained by a more facorable possibility of elastic energy transfer into the surface energy. According to present ideas [4], a crack starts to extend whenever the stress concentration factor reaches a definite constant value which characterizes the strength of a material.

Let us consider the problem of strength of brittle bodies in compression. Let a straight crack be imbedded in a compressive stress field (principal stresses at infinity are equal to N_1 and N_2). Sides of the crack are temporarily considered as stress-free. This somewhat artificial case may be visualized as one of a crack being originally a cavity, the sides of which did not come into contact during compression. For simplicity, we will confine ourselves to the case of plane strain.

Utilising the solution of Muskhelishvili [6], we shall compute the stresses σ_x, σ_y, and τ_{xy}. We assume that the crack of length $2l$, extending along the x-axis with its centre at the origin of Cartesian coordinates xy. We have

$$\sigma_y + \sigma_x = \mathrm{Re}\frac{Az}{\sqrt{z^2 - l^2}} + (N_1 - N_2)\cos 2\alpha$$

$$\sigma_y - i\tau_{xy} = \frac{\bar{A}l^2(z-\bar{z})}{4(\bar{z}^2 - l^2)^{3/2}} - \frac{1}{2}\mathrm{Re}\frac{Az}{\sqrt{z^2-l^2}}, \qquad z = x + iy \qquad (1.1)$$

$$A = N_1 + N_2 - (N_1 - N_2)e^{2i\alpha} \quad (N_1 < 0,\ N_2 < 0)$$

Where α is the angle between the x-axis and the direction of the principal stress N_1.

In the vicinity of the crack tip in its original direction at $z = l + \varepsilon$, where $\varepsilon \ll l$ the stresses (according to (1.1) behave as follows:

$$\sigma_y = \sigma_x = \frac{1}{2}[N_1 + N_2 - (N_1 - N_2)\cos 2\alpha]\sqrt{\frac{l}{2\varepsilon}}$$

$$\tau_{xy} = -\frac{1}{2}(N_2 - N_1)\sin 2\alpha\sqrt{\frac{l}{2\varepsilon}} \qquad (1.2)$$

We will introduce the following basic Hypothesis A. Under conditions of compression, the crack will always be a transverse shear crack (for simplicity, it will be called a shear crack). The beginning of propagation of an equilibrium crack will be determined by a stress-concentration coefficient of the single stress component τ_{xy}, specially since the stresses σ_x and σ_y are compressive.

In case of transverse shear, the critical load causing propagation of the crack along a straight line in accordance with (1.2) is given by the condition [1]

$$\pi\sqrt{l}(N_2 - N_1)\ \sin 2\alpha = 2\sqrt{2L} \qquad (1.3)$$

Where L is the shear cohesion modulus [1], analogous to the cohesion modulus K for normal tensile cracks [4]. L is a constant and describes the resistance of a material to transverse shear.

For comparison, we will also consider the case of the field of tensile stresses N_1 and N_2. If we assume that the crack will be a cleavage crack or, in our terminology, a normal tensile crack extending along a straight line then, by (1.2) and the Barenblatt's condition, the critical load may be found from the equation

$$\pi\sqrt{l}[N_1 + N_2 - (N_1 - N_2)\cos 2\alpha] = 2\sqrt{2}K \quad (N_1 > 0,\ N_2 > 0) \quad (1.4)$$

By considering equations (1.3) and (1.4), the following simple conclusions which are in good agreement with experimental data [7] can be made. Equations (1.3) in particular, imply that among many initial microcracks randomly orientated the most dangerous transverse shear microcracks are those which are oriented at 45° with respect to the planes of principal stresses N_1 and N_2. According to (1.4), the most dangerous normal tensile cracks are those which are parallel to the plane on which the highest tensile stress is applied. From this it follows that, at least at the beginning of fracture, if it is caused by transverse cracks, then the fracture surface will be inclined at 45° to the directions of principal stresses. In case of normal tensile fracture, its surface will coincide with the surface of the maximum principal tension stress. Further, from (1.3) it follows that fracture in compression, like tensile fracture, has an unstable, sudden nature since the crack does not extend initially but it becomes unstable as soon as the critical value of shear stresses is attained.

We will introduce a concept of an isotropic brittle body. We will assume that according to this concept the magnitude of strength of the body is independent of the direction of tension or compression. This is obviously equivalent to the assumption that the most dangerous shear and tearing microcracks always in a macroscopic body during its compression or extension in any direction. Corresponding generalisations for the case of anisotropy do not cause serious difficulties.

In case of uniaxial compression or tension of bodies with the most dangerous cracks, equations (1.3) and (1.4) respectively will have the form:

$$\sigma_+ = \frac{2\sqrt{2}L}{\pi\sqrt{l}}, \qquad \sigma_- = \frac{\sqrt{2}K}{\pi\sqrt{l}} \tag{1.5}$$

Where σ_+ and σ_- are absolute values of strength of an isotropic brittle body in compression and in tension, respectively. From equations (1.5) we obtain:

$$2\frac{\sigma_-}{\sigma_+} = \frac{K}{L} \tag{1.6}$$

from which it follows that for the materials whose tensile strength is considerably higher than their compression strength, a crack in tension will be, as a rule, a normal tearing crack.

The type of crack determines the nature of fracture: tearing or shear. This conclusion may be easily changed for the case of an arbitrary, nonhomogeneous, general field of stress. In particular, it may be shown that one part of the crack occurs by normal tearing, while the other part by transverse shear.

In materials where the compression strength is comparable to tensile strength $\sigma_+ \sim \sigma_-$, a crack may be a transverse shear crack even in a tensile stress field and, it will be propagated at an angle of 45° to the direction of tension. This has been observed for plastic materials [7]. In accordance with (1.3), the tensile strength of such materials will be expressed as:

$$\sigma_{-} = \frac{2\sqrt{2}L}{\pi\sqrt{l}}$$

The hypothesis *A* defines a class of materials whose fracture in compression may be described by the present mechanism. It seems, from experimental data [7 and 3] that this mechanism is best applicable to strong, monolithic brittle materials which break in compression into relatively large pieces. Another mechanism for fracture in compression is possible, as described by the present writer in [8]. This mechanism is connected with a local instability and with local ruptures of the material. It is obviously more characteristic for materials which have a locally non-homogeneous structure. In this case of fracture, a test specimen breaks into a large number of small pieces [8 and 2]. In the latter case, local ruptures may occur both by shear and cleavage mechanisms.

2. An elastic problem for a plane with straight cuts (a plane problem). Let the region occupied by the body be an infinite plane with rectilinear slits $L_k = a_k b_k$ $(k = 1, 2, \ldots, n)$, which are distributed along the straight line which we will take as a real axis Ox. We denote the set of segments L_k as L.

Stresses and displacements in a plane elastic problem may be described [6] by Muskhelishvili's potentials $\Phi(z)$ and $\Psi(z)$, where $z = x + iy$. The following basic relations are valid:

$$\begin{aligned} \sigma_x + i\tau_{xy} &= \Phi(z) + \overline{\Phi(z)} - \overline{\Omega(z)} - (z - \bar{z})\,\overline{\Phi'(z)} \\ \sigma_y - i\tau_{xy} &= \Phi(z) + \overline{\Phi(z)} + \overline{\Omega(z)} + (z - \bar{z})\,\overline{\Phi'(z)} \\ 2\mu\left(\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}\right) &= \varkappa\Phi - \overline{\Phi(z)} - \overline{\Omega(z)} - (z - \bar{z})\,\Phi'(z) \end{aligned} \tag{2.1}$$

Here u, and v are coordinates of the displacement vector in the direction of Cartesian coordinates x and y; σ_x, σ_y, and τ_{xy} are components of the stress tensor; μ and γ are shear modulus and Poisson's ratio, respectively; $\varkappa = 3 - 4\nu$ for the case of plane strain and $\varkappa = (3 - \nu) / (1 + \nu)$ for the case of plane stress.

$$\Omega(z) = z\Phi'(z) + \Psi(z)$$

We assume that at large distances stresses become linear functions of coordinates. This case occurs for example in bending problems or, if the force of gravity is taken into account. Then, near the point at infinity, functions $\Phi(z)$ and $\Omega(z)$ may be written as:

$$\begin{aligned} \Phi(z) &= a_0 z + a_1 + a_2 z^{-1} + O(z^{-2}), \\ \Omega(z) &= b_0 z + b_1 + b_2 z^{-1} + O(z^{-2}) \end{aligned} \tag{2.2}$$

According to (2.1), the coefficients a_0, a_1, b_0, b_1, and b_2 are expressed by means of mechanical quantities as follows:

$$\begin{gathered} a_2 = -\frac{X + iY}{2\pi(1+\varkappa)}, \qquad b_2 = \frac{(\varkappa+1)X - (\varkappa-1)iY}{2\pi(1+\varkappa)}, \qquad a_1 = {}^1/_4\,(N_1 + N_2) \\ b_1 = -{}^1/_2\,(N_1 - N_2)\,e^{-2i\alpha}, \qquad a_0 = {}^1/_4\,[c_1 + c_3 - i\,(c_2 + c_4)], \\ b_0 = {}^1/_2\,(c_3 - c_1) + ic_5 \end{gathered} \tag{2.3}$$

Here (X, Y) denotes the main vector of external forces applied at the sides of the set of slits L ; N_1 and N_2 are values of principal stresses at infinity, α is an angle formed by the axis corresponding to N_1 and by the x-axis. Constants $c_1, c_2, c_3, c_4, c_5,$ and c_6 determine linear behavior of stresses at infinity:

$$\sigma_x = c_1 x + c_2 y, \quad \sigma_y = c_3 x + c_4 y, \quad \tau_{xy} = c_5 x + c_6 y \tag{2.4}$$

When the vector of constant body forces (g_x, g_y) is present, then the following identities obviously hold for the constants $c_1, c_4, c_5,$ and c_6,:

$$c_1 + c_6 + g_x = 0, \qquad c_4 + c_5 + g_y = 0 \tag{2.5}$$

Let us suppose that the interaction between opposite borders of slits is of the type of dry Coulomb friction with coupling and that boundary conditions on slits L may be expressed as follows:

$$\begin{gathered} [v] = \alpha(x), \quad [\sigma_y] = \beta(x), \qquad [\tau_{xy}] = \gamma(x) \\ \tau_{xy}^{\pm} = -k^{\pm}(x) + \rho^{\pm}(x)\,\sigma_y{}^{\pm} \quad \text{on } L \end{gathered} \tag{2.6}$$

Here $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $k^{\pm}(x)$, and $\rho^{\pm}(x)$ are given functions. The parenthesis $([a] = a^+ - a^-)$ denotes the jump in the value of a across the line L ; symbols (+) and (–) denote values on the upper and lower border of slits L, respectively. The symbol $(\pm)$ is used for simultaneous entry of conditions on both the upper and lower borders.

Boundary conditions (2.6) describe a large number of particular cases. For example, when opposite sides of slits which are contiguous with each other exibit relative slipping with a constant coefficient of dry friction ρ and with constant coupling k, then the following relations hold:

$$\begin{gathered} \alpha(x) = \beta(x) = \gamma(x) = 0 \qquad (k > 0,\ \rho > 0) \\ k^+(x) = k^-(x) = k = \text{const}, \qquad \rho^+(x) = {}^-\rho(x) = \rho = \text{const} \end{gathered} \tag{2.7}$$

By using basic Equations (2.1), the boundary conditions (2.6) may be written as follows:

$$\begin{gathered} [\Phi(t) + \overline{\Phi(t)} + \overline{\Omega(t)}] = \beta(t) - i\gamma(t) \quad \text{on } L \\ [(\varkappa + 1)(\Phi(t) - \overline{\Phi(t)}) + \Omega(t) - \overline{\Omega(t)}] = 4\mu i\alpha'(t) \quad \text{on } L \\ \Omega^{\pm}(t) - \overline{\Omega^{\pm}(t)} = -2ik^{\pm}(t) + i\rho^{\pm}(t)\,\overline{(2\Phi^{\pm}(t) + 2\Phi^{\pm}(t) + \Omega^{\pm}(t) + \Omega^{\pm}(t))} \end{gathered} \tag{2.8}$$

From the first two boundary conditions of (2.8) we can see that:

$$\begin{gathered} [\operatorname{Im}\Omega(t)] = \gamma(t), \quad [\operatorname{Im}\Phi(t) = \frac{1}{\varkappa + 1}(2\mu\alpha'(t) - \gamma(t)) \\ [Re\,(2\Phi(t) + \Omega(t))] = \beta(t) \quad \text{on } L \end{gathered} \tag{2.9}$$

From (2.9) it follows that the analytic function $2\Phi(z) + \Omega(z)$ exibits a discontinuity

$f(t)$ across the slits L:

$$[2\Phi(t) + \Omega(t)] = f(t), \qquad f(t) = \beta(t) + \frac{i}{\varkappa+1}(4\mu\alpha'(t) + (\varkappa - 1)\gamma(t)) \tag{2.10}$$

Hence, by employing Conditions (2.2) at infinity and according to the Sokhotskii-Plemeli formula we obtain:

$$2\Phi(z) + \Omega(z) = \frac{1}{2\pi i}\int_L \frac{f(t)\,dt}{t-z} + (2a_0 + b_0)z + 2a_1 + b_1 \tag{2.11}$$

By (2.9) and (2.11), the last boundary condition (2.8) may be written as

$$\Omega^{\pm}(t) - \overline{\Omega^{\pm}(t)} = g^{\pm}(t) \text{ on } L$$

$$g^{\pm}(t) = -2ik^{\pm}(t) + i\rho^{\pm}(t)\Big\{\pm\beta(t) + \tag{2.12}$$

$$+\frac{1}{\pi}\int_L\left[\gamma(\tau)\frac{\varkappa-1}{\varkappa+1} + \frac{4\mu\alpha'(\tau)}{\varkappa+1}\right]\frac{d\tau}{\tau - t} + 4a_1 + 2\mathrm{Re}\,b_1 + 2t(2\mathrm{Re}\,a_0 + \mathrm{Re}\,b_0)\Big\}$$

Solution of the Dirichlet problem (2.12) for the outer parts of slits distributed along one and the same straight line is treated in the monographs of Muskhelishvili [9] and Gakhov [10]. Finally, taking acount of conditions at infinity (2.2), the solution of the boundary value problem (2.12) in the class of function which are unbounded (but integrable) at the ends of slits L_k is

$$\Omega(z) = \frac{1}{4\pi i X(z)}\int_L \frac{[g^+(t) + g^-(t)]\,X(t)}{t-z}\,dt +$$

$$+\frac{1}{4\pi i}\int_L \frac{g^+(t) - g^-(t)}{t-z}\,dt + \frac{P_{n+1}(z)}{X(z)} + \frac{1}{2}(b_0 + \bar{b}_0)z + \frac{1}{2}(b_1 + \bar{b}_1) \tag{2.13}$$

Here $P_{n+1}(z)$ is a polynomial of the $(n+1)$ order with purely imaginary coefficients.

$$P_{n+1}(z) = C_0 z^{n+1} + C_1 z^n + \ldots + C_{n+1}, \qquad X(z) = \prod_{k=1}^{n}(z - a_k)^{1/2}(z - b_k)^{1/2} \tag{2.14}$$

We assume that the function $X(z)$ for large z becomes

$$X(z) = z^n + \alpha_{n-1}z^{n-1} + O(z^{n-2}) \tag{2.15}$$

While $X(t)$ is the value of $X(z)$ at the upper surface of slits L.

It remains to determine the coefficients of the polynomial $P_{n+1}(z)$. The coefficients c_0 and c_1 are readily found from conditions at infinity (2.2), which after expansion of $\Omega(z)$ about the point at infinity and comparison of coefficients of z of the first and zero power give

$$C_0 = i\,\mathrm{Im}\,b_0, \qquad C_1 = i\,\mathrm{Im}\,b_1 + i\alpha_{n-1}\mathrm{Im}\,b_0 \tag{2.16}$$

The remaining n coefficients may be found from n conditions which express the fact that the discontinuities in both, v and dv/dx are given across the slits. These conditions may be written as a system of n linear equations in terms of the unknowns C_2, C_3 . . . , C_{n+1} and it may be shown that the solution of the system is unique (in a way which is entirely analogous to that used in the book of Muskhelishvili [6]). If the vector of external forces (X, Y) is given, two of these conditions may be obtained simply from the expansion at infinity.

Note. The described method of solution of the elastic problem (2.6) leads also to an effective closed solution of a seemingly much more difficult nonlinear problem, when the interaction of forces between the opposing slit edges is of arbitrary nature

$$\tau_{xy} = F(\sigma_y) \tag{2.17}$$

where $F(x)$ is an arbitrary function. At the same time, boundary conditions of the elastic problem may be given in the form

$$[v] = \alpha(x), \quad [\sigma_y] = \beta(x), \quad [\tau_{xy}] = \gamma(x), \quad \tau_{xy} = F(\sigma_y) \quad \text{on } L \tag{2.18}$$

It is easy to see that the solution of the last problem (2.18) will be expressed by the equation (2.11) and by (2.13) in which $g^{\pm}(t)$ is

$$g^{\pm}(t) = F\{V^{\pm}(t)\}$$

$$V^{\pm}(t) = \operatorname{Re}\left\{\pm\frac{1}{2}f(t) + \frac{1}{2\pi i}\int_L \frac{f(\tau)\,d\tau}{\tau - t} + (2a_0 + b_0)t + 2a_1 + b_1\right\} \tag{2.19}$$

Here, as previously, the principal value of the integral is considered. Let us emphasize that the expression (2.11) is solely the result of conditions of discontinuities in normal displacement and also in normal and tangential stresses on L.

3. Strength of brittle bodies in compression. We will assume that a real brittle body contains a large number of randomly distributed microcracks or slits. It is impossible (and unnecessary) to determine the interaction between these microcracks exactly, hence the following approximate approach will be employed.

We will choose one arbitrary microcrack and 'blur' all other microcracks throughout the body, replacing their influence on the chosen microcrack by the influence of an elastic body which we shall now consider homogeneous. This body will have average elastic constants and will contain no cracks (except for the chosen one). Let us consider the stress problem in the vicinity of the chosen crack.

Thus, let us consider a straight crack which is plane deformed in a stress field of two principal compression stresses N_1 and N_2. The crack of the length $2l$ lies along the x-axis with its center at the origin of Cartesian coordinates xy. Opposite sides of the crack touch each other at every point due to compression loading so that the crack in this case represents a line of discontinuity of the tangential displacement u only. The displacement component v, normal to the crack surface is not discontinuous. Equilibrium considerations require also continuous stresses σ_y and τ_{xy}. Another condition should be added which would describe the interaction of the opposite sides of the crack when they touch each other.

This condition will be assumed as the one of dry Coulomb friction with coupling. The coupling constant k and the friction coefficient are assumed constant along the entire crack ($k > 0,\ \rho > 0$).

It should be noted that condition of dry Coulomb friction with coupling is a more or less good representation of physical conditions of the real problem only if the opposite surfaces of the crack slide along each other. If such sliding does not take place (e.g., if the crack lies in a plane of the principal compressive stress), this condition becomes unacceptable. Finally, the boundary conditions are written as

$$|x| < l \quad \text{when} \quad y = 0$$
$$[v] = 0,\ \ [\sigma_y] = 0,\ \ [\tau_{xy}] = 0,\ \ \tau_{xy} = -k + \rho\sigma_y \tag{3.1}$$

Due regard is given to the sign of the stress σ_y in the last condition of (3.1) (also, the direction of shear is given).

The elastic problem (3.1) is a particular case of the general problem (2.6). Obviously the following relations hold:

$$\begin{gathered}\alpha(x) = \beta(x) = \gamma(x) = 0,\ \ k(x) = k,\ \ \rho(x) = \rho,\ \ f(x) = 0,\ \ a_0 = b_0 = 0\\ a_2 = b_2 = 0\\ g^{\pm}(t) = g = -2ik + i\rho\,[N_1 + N_2 - (N_1 - N_2)\cos 2\alpha]\\ C_0 = C_3 = 0,\quad C_1 = {}^1\!/_2 i(N_1 - N_2)\sin 2\alpha\ (N_1 < 0,\ N_2 < 0)\end{gathered} \tag{3.2}$$

In the particular case (3.2), the formulas (2.11) and (2.13) will become

$$2\Phi(z) + \Omega(z) = 2a_1 + b_1, \qquad \Omega(z) = \frac{(2C_1 - g)\,z}{2\sqrt{z^2 - l^2}} + \frac{1}{2}g + \operatorname{Re} b_1 \tag{3.3}$$

We will now consider the state of stress in the vicinity of the end of the crack at $z = l + \varepsilon$, where $\varepsilon = re^{i\theta}(r \ll l)$ When $z \to l$, functions $\Phi(z)$ and $\Omega(z)$ become

$$\Omega(z) = -2\Phi(z) = \frac{(2C_1 - g)\sqrt{l}}{2\sqrt{2r}}\,e^{-{}^1\!/_2 i\theta} + O(1) \tag{3.4}$$

(r and θ are polar coordinates with their centre at $z = l$). Hence, according to (2.1) the stresses at the crack tip may be written as:

$$\begin{gathered}\sigma_x + i\tau_{yx} = Bi\sqrt{{}^1\!/_2\, l / r}\ \ (3 - e^{-i\theta} - i\sin\theta\cdot e^{i\theta})e^{{}^1\!/_2 i\theta}\\ \sigma_y - i\tau_{yx} = Bi\sqrt{{}^1\!/_2\, l / r}\,(-1 - e^{-i\theta} + i\sin\theta\cdot e^{i\theta})e^{{}^1\!/_2 i\theta}\\ B = {}^1\!/_4\,[2k - \rho(N_1 + N_2) + (N_1 - N_2)(\rho\cos 2\alpha + \sin 2\alpha)]\end{gathered} \tag{3.5}$$

According to (3.5), the stress components are expressed in polar coordinates r and θ by formulas:

$$\begin{gathered}\sigma_\theta + \sigma_r = \sigma_x + \sigma_y = -4B\sqrt{{}^1\!/_2 l / r}\,\sin {}^1\!/_2\theta\\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} = 2B\sqrt{{}^1\!/_2 l / r}\ ie^{{}^3\!/_2 i\theta}(2 - i\sin\theta e^{-i\theta})\end{gathered} \tag{3.6}$$

In particular, the stress components along the real axis i.e. when $y = 0$, are

$$\begin{aligned}
\sigma_y &= \sigma = {}^1/_2\,[N_1 + N_2 - (N_1 - N_2)\cos 2\alpha] \\
\tau_{xy} &= \begin{cases} -k + \rho\sigma & \text{for } |x| < l \\ -k + \rho\sigma + 2B\,|x| / \sqrt{x^2 - l^2} & \text{for } |x| > l \end{cases} \\
\sigma_x &= \tau = {}^1/_2\,[N_1 + N_2 + (N_1 - N_2)\cos 2\alpha] \quad \text{for } |x| > l \\
\sigma_x &= \tau - 4Bx / \sqrt{l^2 - x^2} \quad \text{on the upper surface for } |x| < l \\
\sigma_x &= \tau + 4Bx / \sqrt{l^2 - x^2} \quad \text{on the lower surface for } |x| < l
\end{aligned} \tag{3.7}$$

Obviously, the stress σ_y along the crack and in the direction following it is constant and equal to the same stress σ_y which would exist at every point of the body if the crack were absent. This is easily seen to be true for any number of cracks which are distributed along one and the same straight line. This fact is also independent of the nature of interaction between opposite sides of cracks and follows only from conditions of continuity of normal displacement, shear stress and of normal stress.

The stress σ_x is constant on the line of extension of the crack. At $|x| < l$, near to crack tips, the stress σ_x tends to be infinite, the sign being different for the two ends along one side and also for two sides at one and the same crack tip. Finally, we introduce a formula describing shear stress concentration at a tip of the crack and on its line of extension:

$$\tau_{xy} = 2B\sqrt{{}^1/_2\, l / r} + O(1) \qquad \text{for } y = 0, \quad x = l + r \tag{3.8}$$

We now make use of the hypothesis A : a crack under compression will always be a transverse shear crack.

Introduction of this hypothesis is justified by the fact that in the vicinity of a crack tip and on its line of extension, stresses σ_y and σ_x do not have singularities and only the stress τ_{xy} tends to infinity, in accordance with (3.8). It is natural to assume, therefore, that the onset of crack propagation is determined only by the concentration coefficient of the shear stress τ_{xy}. Simple computations, according to (3.6), further show that τ_{xy} is the maximum shear stress at the points of the body near the tips of the crack and on its line of extension. The critical load follows from the condition [1]

$$2\pi B\sqrt{l} = -L\sqrt{2} \qquad (L > 0) \tag{3.9}$$

We shall now determine distribution of the most dangerous transverse shear cracks. These are obviously the cracks at which the critical value of the shear stress concentration coefficient is attained first, for otherwise equal conditions. This means that the orientation of the most dangerous cracks is defined by the angle which minimises the modulus of the function $B(\alpha)$. Analysis of the function $B(\alpha)$, given by (3.5), shows that depending on the direction of relative shear of the opposite sides of the crack and of the relationship between N_1 and N_2, the most dangerous cracks may lie at the angle of $\frac{1}{2}\cot^{-1}\rho$ or at the angle of $\frac{1}{2}\pi + \frac{1}{2}\cot^{-1}\rho$ with the direction of N_1. For example, for $\rho = 0.3$, the value of $\frac{1}{2}\cot^{-1}\rho$ is 37°. In this case, if the friction coefficient ρ is equal to zero, which corresponds to the case of coupling (adhesion) friction between opposite surfaces of the crack only, the most dangerous shear cracks will be distributed at the angle 45° to the planes of principal stresses. The last result is identical with a corresponding result obtained from the simplified theory.

In case of uniaxial compression of a body containing the most dangerous crack, the condition (3.9) will become

$$\sigma_+ = \frac{2}{\sqrt{1+\rho^2}-\rho}\left(k + \frac{\sqrt{2}L}{\pi\sqrt{l}}\right) \quad (3.10)$$

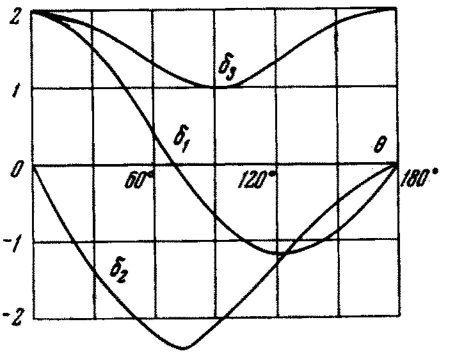

FIG. 1

The formula (3.10) defines strength of an isotropic brittle body in compression as a function of the shear cohesion modulus L, of the coupling (adhesion) constant k, of the friction coefficient ρ and of the original length l. The higher the magnitude of k, L and ρ, the higher the compression strength σ_+ of the body. The shorter the original crack length, the higher the strength σ_+. Equation (3.10) shows also that as soon as the load attains the limiting value σ_+, the equilibrium of a body containing a crack $2l$ long becomes unstable and further increase in load leads to dynamical growth of the crack.

In case of ideally smooth surfaces ($k = 0$, $\rho = 0$) the compression strength of a body equals the value of strength which was obtained from the simplified theory (first equation of (1.5)).

It is of interest to compare the moduli of cohesion L and K. By using the second formula of (1.5) and the relation (3.10) we obtain:

$$\frac{L}{K} = \frac{\sigma_+}{\sigma}\left(\frac{\sqrt{1+\rho^2}-\rho}{2} - \frac{k}{\sigma_+}\right) \quad (3.11)$$

4. An arbitrary 'closed' crack. Let us consider a brittle body of arbitrary shape under some loading, which contains an arbitrary curvilinear crack such that its opposite sides touch each other at the end of the crack (say, because of the local compression). Let us choose the origin of polar coordinates $r\theta$ so as to coincide with the tip of the crack in question ($\theta = 0$ corresponds to the direction of the crack extention). It may be shown, by using the 'microscope' principle and the formula (3.6), that stresses σ_r, σ_θ, $\tau_{r\theta}$ and τ_{max} behave close to the crack end as follows:

$$\sigma_\theta = B(p,l)\delta_2(\theta)r^{-1/2}, \qquad \tau_{r\theta} = B(p,l)\delta_1(\theta)r^{-1/2}$$

$$\tau_{max} = B(p,l)\delta_3(\theta)r^{-1/2}, \qquad \sigma_\theta + \sigma_r = -4B(p,l)r^{-1/2}\sin^1/_2\theta \quad (4.1)$$

Where

$$\delta_1(\theta) = 2\cos{}^3/_2\,\theta + \sin\theta\sin{}^1/_2\,\theta$$

$$\delta_2(\theta) = -2\,[\sin{}^3/_2\,\theta + (\sin{}^1/_2\,\theta)^3], \qquad \delta_3(\theta) = \sqrt{{}^5/_2 + {}^3/_2\cos 2\theta}$$

The function $B(p, l)$ depends only on the external load and on the shape of the body. Functions $\delta_1(\theta)$, $\delta_2(\theta)$, and $\delta_3(\theta)$ are shown in Fig. 1.

It is obvious that all stresses (and any homogeneous combination of stresses) at the end of the crack may be written as a product of a function $B(p, l)r^{-1/2}$, which remains the same for all stresses and of some function θ, which is characteristic for a given stress (or combination of stresses). This remarkable circumstance follows only from conditions of contact of opposite sides of the crack at its ends (i.e. from conditions of continuity of the normal displacement, normal stress and shear stress). The remaining boundary condition may have an arbitrary character and be more general than (2.17).

This property, which is typical for 'closed' cracks, is rather distinctive in comparison with the case of the crack with load-free sides at the end. It enables formulation of the following conclusion.

The behavior of a 'closed' crack is completely self-governing, i.e. the direction of crack propagation and the nature of fracture (and also the limiting concentration coefficient) do not depend on the shape of the body and on external loads but are fully defined by material properties at the crack end. This argument will be called the Autonomy Law. Cracks considered earlier were assumed to propagate along a straight line or were forced to propagate that way (e.g. by cementing). The Autonomy Law allows us to analyze this assumption. According to the Autonomy Law, the crack propagation direction and nature of fracture are fully defined by the functions $\delta_1(\theta)$, $\delta_2(\theta)$, and $\delta_3(\theta)$ (or by some other functions provided that they themselves are completely defined).

Let us consider the curves of Fig. 1. It can be seen that the shear stress $\tau_{r\theta}$, is an even function of θ and has three extremes, one at $\theta = 0°$ ($\delta_1 = 2$) and two at $\theta = \pm 124°$ ($\delta_1 = -1.27$). Normal stress σ_θ is an odd function of θ and has, similarly, two extremes of opposite signs, at $\theta = \pm 75°$ ($\delta_2 = \mp 2.32$).

Analysis of these curves enables us, to explain the basic tendencies of crack propagation and the very mechanism of fracture itself, in terms of the Autonomy Law. Fracture may occur by means of the shear mechanism, and the crack always propagates in the direction of the highest shear stress $\tau_{r\theta}$ i.e. in the direction of the extended crack line $\theta = 0$, as soon as the absolute magnitude of $B(p, l)$ attains the limiting value

$$|B(p, l)| = \frac{1}{2\pi} L \tag{4.2}$$

Such cracks are always straight, in agreement with the Autonomy Law, since the above theory refers to straight cracks.

Fracture may also occur by means of the tear mechanism, because in the region $0 < \theta < \pi$, or at $-\pi < \theta < 0$ a tensile stress σ_θ always exists and the normal tearing crack always spreads from the end $r = 0$ in the direction of maximum σ_θ, i.e. at the angle of $\theta = +75°$, or $\theta = -75°$ as soon as the modulus of the magnitude $B(p, l)$ attains the limiting value

$$B(p, l)| = 0.14K \tag{4.3}$$

The normal tearing crack has, in the beginning, load-free surfaces since it lies in a tensile stress region. During subsequent propagation, it enters inevitably the region of compressive stresses, and becomes a 'closed' crack, i.e. becomes subject to the Autonomy Law. The crack then has two alternatives: it either propagates in a straight line (shear) or deviates by $+75^\circ$ or -75° (tearing) from the original direction. Depending on the final outcome, the crack may finish with a completely arbitrary shape ranging from a straight line to a steplike or 'sprucelike' curve.

Symmetry considerations and the Autonomy Law indicate that in an ideally homogeneous material and in a homogeneous field of stress, a crack will either have a strongly periodic structure or will be more or less straight (if the shape of original defect is neglected).

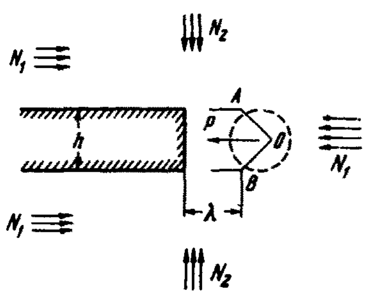

FIG. 2

The 'quantum' character of the direction of propagation discussed above, represents a significant feature of brittle cracks in compressed solid.

Thus, cracks in compressed bodies generally possess a stepwise structure with known direction of individual steps; the length of each step depends on the elastic stress field present in the body.

A mathematical solution of the elastic problem of stepwise cracks is rather complicated. However, when the characteristic length of one step is small in comparison with the entire length of the crack, the stepwise structure will be just a fine structure of some average smooth crack. Such a large-scale smooth crack may be again analyzed by the method described above, which leads to similar results. Then, the elementary volume should be larger in comparison with the characteristic size of the fine step. In particular, for large cracks a generalized Autonomy Law holds which indicates that the macrostructure of the crack has, generally speaking, a stepwise structure with a specific direction of separate steps (0° and 75°). When the body is sufficiently large, a sequence of stepwise structures differing in size may be generally observed.

5. The rock burst. The rock burst is one of the most challenging cases for the application of the theory of crack propagation on compressed bodies. The rock burst is a sudden outburst of matter into the excavated space without the release of gas. The higher the rock pressure and strength of material is [2 and 3], the more dangerous it becomes.

The effect may be visualized using a simple theoretical picture. Let us assume that in a homogeneous, isotropic brittle body there is a horizontal excavation of height h (Fig. 2). The excavation is of a usual shape which is close to a rectangular one and is within a pressure field generated by the rock (N_1 is the horizontal, N_2 is the vertical pressure).

An analysis of elastic stresses shows that in front of the excavation at a distance of about (1 to 2) h there is an interior focus of concentration of the specific elastic energy and of the compression stresses σ_x (shown by the dashed circle in Fig. 2). It is within this region, that the shear cracks AO and OB originating at the sides of the most dangerous natural cracks, begin to form and propagate.

During this period which precedes the rock burst a crack forms at the site of unstable shear cracks, and continues to grow in certain directions, until a stable size which is of order of the region of concentration, i.e. of order h is reached. Cracks AO and OB form a wedge which is pressed out by rock pressure. The magnitude of the force P acting on the wedge is approximately equal to

$$P = {}^1/_2\,(\beta_1 N_1 + \beta_2 N_2)\,h \tag{5.1}$$

where β_1 and β_2 are stress concentration factors. Thus, the compression strength σ_+ plays a decisive role during the first period. The second period is characterized by catastrophic propagation of cracks from points A and B towards the corners of the excavation other foci of stress concentration being present at these corners, and by subsequent outburst of the matter.

In the instant preceding the outburst, the force P is in equilibrium with the resistance of the material of the rock, which is equal to $2\sigma_s\lambda$ (σ_s is the shear strength of the material). Hence we have the relation

$$\beta_1 N_1 + \beta_2 N_2 = 4\,\frac{\lambda}{h}\,\sigma_s \tag{5.2}$$

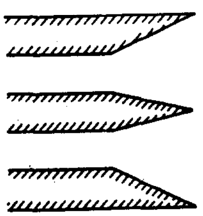

FIG. 3

Immediately before the second period there occurs a redistribution of stresses leading to an increase of P and decrease in λ (the excavation moves forwards).

Thus, a rock burst is the result of action of the internal focus of concentration of specific internal energy and of compression stresses in front of the excavation. A change in the shape of the excavation consisting in subsequent transition into a cone would remove the danger of a strong rock burst because then the focus of stress concentration would form at the cusp and all fractures would be accompanied by an instant decomposition of mater so that elastic energy would not concentrate. As an example, forms of excavations shown in Fig. 3 are safer than the usual form. (An ideally safe form would be the form of natural shear).

Finally, we consider the problem of modeling of the rock burst. As shown earlier, the decisive parameters are L, K, σ_+, and σ_s. We shall denote the characteristic linear dimension by d. The magnitude of rock pressure will be denoted by p (e.g., $N_2 = p$, $N_1 = \gamma p$, $\gamma \leqslant 1$). According to the π – theorem:

$$\frac{p}{\sigma_s} = \Gamma\left\{\frac{L}{\sigma_s\sqrt{d}}, \frac{\sigma_+}{\sigma_s}, \frac{K}{\sigma_s\sqrt{d}}, \nu\right\} \tag{5.3}$$

The function Γ has to be determined from experiments with models.

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